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Quantum heat engine beyond the adiabatic approximation

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Abstract

We introduce a new quantum heat engine, in which the working medium undergoes quantum quasi-adiabatic and non-unitary processes to extract work in the cycle. Two kinds of working medium are considered. The first one consists of a three-level quantum system, whereas the second is a quantum system with a discrete level and a continuum. Net work done by this engine is calculated and discussed. The results show that this quantum heat engine behaves like the two-level quantum heat engine in both the high-temperature and low-temperature limits, but it operates differently at intermediate temperatures. The efficiency of this quantum heat engine is also presented and discussed.

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1. Introduction

A classical heat engine converts heat energy into mechanical work by using a classical-mechanical system in which a working medium (for example, a gas) expands and pushes a piston in a cylinder. Working between a high-temperature reservoir and a low-temperature reservoir, the classical heat engine achieves maximum efficiency when it is reversible, while the efficiency is zero if the two reservoirs have the same temperature. The situation changes for its quantum counterpart, where the working medium and the dynamics that govern the cycle are quantum. It was shown that the quantum heat engine can better the work extraction and improve the engine efficiency [1–5].

The quantum heat engine concept was introduced by Scovil and Schulz-Dubois [6] and extended in many later works [1–5, 7–12]. Quantum heat engines are characterized by three attributes: the working medium, the cycle of operation and the dynamics that govern the cycle. In the previous works, the working medium is considered as an ensemble of many non-interacting discrete level systems. Specifically, the analysis is carried out on two-level systems

[2, 13], three-level systems [4], as well as an ensemble of harmonic oscillators [2, 14, 15]. In these analyses, the occupation probabilities of the energy levels are assumed unchanged in adiabatic processes, i.e. the evolution of the working medium is unitary and adiabatic. These give rise to the following questions: how does the work extraction change when the quantum adiabatic conditions break down? With continuum working medium what is the work extraction of quantum heat engine? Can such a quantum heat engine improve the work extraction?

In this paper, we will answer these questions by examining a quantum heat engine working between two reservoirs with different temperatures. Two kinds of working medium are considered. The first one is modelled as a three-level quantum system and the second is described by a quantum system with a discrete level and a continuum. In contrast with the previous studies, we consider in this paper a quantum quasi-adiabatic process instead of the quantum adiabatic process in the literatures, and the evolution is not necessary unitary, resulting from couplings of the working medium to its environment. The quantum quasi-adiabatic process here means that the working medium undergoes a quantum evolution with only the population of the ground state unchanged. This is of relevance to the fast change of controlled parameters upon which the energy levels of the working medium depend. So, the presented study put forward the research in this field by reconsidering two attributes among the three: the working medium and the cycle of operation. The work extraction and the efficiency of this new quantum heat engine are presented and discussed. The results show that the quantum quasi-adiabatic and non-unitary process affects the work extraction and can increase the efficiency of the quantum heat engine.

This paper is organized as follows. In section 2, we calculate the work extraction and the efficiency of the quantum heat engine with a three-level quantum system as its working medium. The positive work condition is given and its relation to other quantum heat engine is discussed. We extend the discussion for the three-level quantum heat engine to the quantum heat engine in which the working medium is a quantum system with a discrete level and a continuum in section 3. Finally, we conclude our results in section 4.

2. Three-level quantum heat engine working beyond the quantum adiabatic approximation

In this section, we shall present a calculation for the work extraction and the efficiency of the three-level quantum heat engine. The three levels of the working medium at stage S are labelled by E_0^S, E_1^S and E_2^S $\left(E_2^S > E_1^S > E_0^S\right)$, respectively. The heat-engine cycle includes the following four stages. In stage 1, the three-level system contacts with a heat bath at temperature T^h. After some time, the system has probability q_0^h in its ground state E_0^h, satisfying

$$1 - q_0^h = \frac{1}{z_{hh}}\left(\mathrm{e}^{-\beta_h E_1^h} + \mathrm{e}^{-\beta_h E_2^h}\right), \tag{1}$$

where z_{hh} denotes the partition function of the system at this stage. In stage 2, quasi-adiabatic changes of the energy levels from $\left\{E_0^h, E_1^h, E_2^h\right\}$ to $\left\{E_0^l, E_1^l, E_2^l\right\}$ take place. The quasi-adiabatic changes mean that only the population of the ground state keeps constant, i.e. after the change one has

$$1 - q_0^h = \frac{1}{z_{hl}}\left(\mathrm{e}^{-\beta_h E_1^l} + \mathrm{e}^{-\beta_h E_2^l}\right). \tag{2}$$

This evolution is not unitary that may be described by the master equation,

$$\mathrm{i}\frac{\partial}{\partial t}\rho = [H, \rho] + \mathcal{L}(\rho), \tag{3}$$

where ρ describes the density matrix of the working medium initially in

$$\rho(0) = q_0^h|E_0\rangle\langle E_0| + \frac{\mathrm{e}^{-\beta_h E_1^h}}{Z_{hh}}|E_1\rangle\langle E_1| + \frac{\mathrm{e}^{-\beta_h E_2^h}}{Z_{hh}}|E_2\rangle\langle E_2|. \tag{4}$$

$\mathcal{L}(\rho)$ comes from the working-medium–environment coupling. The external sources of work [16, 17] may play the role of the environment. This is different from [16, 17], where the time evolution of the working medium in this stage is unitary. To be specific, $\mathcal{L}(\rho)$ may take

$$\begin{aligned}\mathcal{L}(\rho) = {} & \mathrm{i}\Gamma_1(2|E_1\rangle\langle E_2|\rho|E_2\rangle\langle E_1| - \rho|E_2\rangle\langle E_2| - |E_2\rangle\langle E_2|\rho) \\ & + \mathrm{i}\Gamma_2(2|E_2\rangle\langle E_1|\rho|E_1\rangle\langle E_2| - \rho|E_1\rangle\langle E_1| - |E_1\rangle\langle E_1|\rho).\end{aligned} \tag{5}$$

Γ_1 and Γ_2 depend on the temperature of the external source of work (environment) though the working medium was isolated from the heat bath. We assume that $H = H(\vec{R})$ changes slowly such that the population in state $|E_0\rangle$ remains unchanged, while a population transfer among $|E_1\rangle$ and $|E_2\rangle$ occurs in this stage and finally the population on these states after stage 2 are $\frac{1}{z_{hl}}\,\mathrm{e}^{-\beta_h E_1^l}$, and $\frac{1}{z_{hl}}\,\mathrm{e}^{-\beta_h E_2^l}$, respectively. z_{hl} is the partition function. This implies that the level E_1 acquires a population gain $\left(\mathrm{e}^{\beta_h E_1^l}/z_{hl} - \mathrm{e}^{\beta_h E_1^h}/z_{hh}\right)$ in this quasi-adiabatic process; this can be done in principle by manipulating the change rate of the parameters on which the energy level depends. The diagonal matrix element $\rho_{ii} = \langle E_i|\rho|E_i\rangle$ represents the probability of the working medium in the state $|E_i\rangle$. It is easy to check that $\rho_{22}/\rho_{11} = \Gamma_2/\Gamma_1$ in the stationary state, i.e. $\langle E_i|\dot{\rho}|E_i\rangle = 0$. Usually, Γ_2 and Γ_1 respectively take $\gamma\bar{n}$ and $\gamma(\bar{n}+1)$, leading to $\rho_{22}/\rho_{11} = \exp[-\beta_h(E_2 - E_1)]$ when $\bar{n} = 1/(\mathrm{e}^{\beta_h(E_2-E_1)} - 1)$. Here E_i depends on $\vec{R}$ and $E_i(\vec{R}_0) = E_i^h$, $E_i(\vec{R}_\tau) = E_i^l$, where τ is the time when stage 2 finishes. Please note that the term $\mathcal{L}(\rho)$ does not directly depend on the population of the state $|E_0\rangle$. So, the population q_0^h in the state $|E_0\rangle$ remains unchanged provided $H(\vec{R})$ changes slowly such that the transition probability induced by the change of $H(\vec{R})$ is negligible. Loosely speaking, the transition probability from the state $|E_0\rangle$ to the state $|E_i\rangle$ induced by the change of $H(\vec{R})$ is proportional to $\left|\frac{\langle E_0|\partial H(\vec{R})/\partial t|E_i\rangle}{(E_0-E_i)^2}\right|$, so if $\left|\frac{\langle E_0|\partial H(\vec{R})/\partial t|E_i\rangle}{(E_0-E_i)^2}\right| \ll 1$ for either $i = 1$ and $i = 2$, the population q_0^h would remain unchanged in the stage. For recent progress in adiabatic evolution, we refer the reader to [18]. In stage 3, the system is brought into contact with another heat bath at temperature T^l, and after some time the population on the ground state E_0^l is q_0^l,

$$1 - q_0^l = \frac{1}{z_{ll}}\left(\mathrm{e}^{-\beta_l E_1^l} + \mathrm{e}^{-\beta_l E_2^l}\right). \tag{6}$$

In stage 4, similar to stage 2, the working medium undergoes another non-unitary and non-adiabatic evolution. Finally, the system couples to the heat bath and relax to the canonical state at temperature T_h; this is also the first stage of the next cycle. By the same procedure [2, 4, 9], the net work done in the cycle is calculated according to $đW = \sum_i p_i\,\mathrm{d}E_i$,

$$\Delta W = \left(q_0^l - q_0^h\right)\left(\Delta_3^h - \Delta_3^l\right) + \left(\delta_3^h - \delta_3^l\right)\left(\frac{1-q_0^h}{1+\mathrm{e}^{\frac{\delta_3^h}{KT_h}}} - \frac{1-q_0^l}{1+\mathrm{e}^{\frac{\delta_3^l}{KT_l}}}\right), \tag{7}$$

where $E_1^x - E_0^x = \Delta_3^x$, $E_2^x - E_1^x = \delta_3^x (x = l, h)$. Clearly, if $\delta_3^h = \delta_3^l$, the work extraction reduces to

$$\Delta W = \left(q_0^l - q_0^h\right)\left(\Delta_3^h - \Delta_3^l\right). \tag{8}$$

This is exactly the work extraction of the two-level quantum heat engine. Furthermore, in the small δ_3^x limit, i.e.

$$\frac{\delta_3^h}{KT_h} \ll 1, \qquad \frac{\delta_3^l}{KT_l} \ll 1, \tag{9}$$

the work extraction reads

$$\Delta W = \left(q_0^l - q_0^h\right)\left(\left(\Delta_3^h + \frac{\delta_3^h}{2}\right) - \left(\Delta^l + \frac{\delta_3^l}{2}\right)\right). \tag{10}$$

This is different from the three-level quantum heat engine with adiabatic processes [4]. The difference can be explained as the non-zero heat exchange in the quantum quasi-adiabatic process. Now, we turn to study the efficiency of this quantum heat engine. It is defined as the ratio of the work done to the heat absorbed in the cycle,

$$\eta = \frac{\Delta W}{\Delta Q}. \tag{11}$$

By the first law of thermodynamics, we have

$$\eta = 1 - \frac{\Delta Q_2 + \Delta Q_3}{\Delta Q_1 + \Delta Q_4}, \tag{12}$$

where $\Delta Q_i (i = 1, 2, 3, 4)$ denote the heat exchange between the working medium and reservoirs on the branch i. By the definition $đQ = \sum_i E_i \mathrm{d}p_i$, simple calculation shows that

$$\eta = 1 - \frac{\left(q_0^h - q_0^l\right)\Delta_3^l + \delta_3^l\left(\frac{1-q_0^h}{1+\mathrm{e}^{\frac{\delta_3^h}{KT_h}}} - \frac{1-q_0^l}{1+\mathrm{e}^{\frac{\delta_3^l}{KT_l}}}\right)}{\left(q_0^h - q_0^l\right)\Delta_3^h + \delta_3^h\left(\frac{1-q_0^h}{1+\mathrm{e}^{\frac{\delta_3^h}{KT_h}}} - \frac{1-q_0^l}{1+\mathrm{e}^{\frac{\delta_3^l}{KT_l}}}\right)}. \tag{13}$$

In the small δ_3^x limit, i.e.

$$\frac{\delta_3^h}{KT_h} \ll 1, \qquad \frac{\delta_3^l}{KT_l} \ll 1, \tag{14}$$

the efficiency becomes

$$\eta = 1 - \frac{\Delta_3^l + \frac{\delta_3^l}{2}}{\Delta_3^h + \frac{\delta_3^h}{2}}. \tag{15}$$

When $q_0^h = q_0^l$, we have

$$\eta = 1 - \frac{\delta_3^l}{\delta_3^h}. \tag{16}$$

This is same as in the conventional three-level quantum heat engine in the limit $q_0^h = q_0^l$. Equation (12) shows that $\eta = 1 - \Delta^l/\Delta^h$ with $\delta_3^l = \delta_3^h = 0$; this is exactly the efficiency of the two-level quantum engine. For $\delta_3^h = 0$ and $\delta_3^l \neq 0$, $\eta = 1 - \left(\Delta_3^l + \delta_3^l/2\right)\big/\Delta_3^h > \Delta^l/\Delta^h$ in the small δ_3^x limit, suggesting that we may increase the efficiency of the engine by manipulating δ_3^h and δ_3^l. This quasi-adiabatic heat engine works with at least a three-level working medium, because it requires that the population in one of the levels remains unchanged, while population among the other levels allowed. The quantum heat engine with a continuum working medium has a similar property, as one can see in the following section.

3. Quantum heat engine with the continuum working medium

In this section, the working medium is envisioned as a quantum system with a discrete level $|d\rangle$ and a continuum $|c\rangle$ as shown in figure 1. The heat-engine cycle consists of four stages labelled by 1, 2, 3 and 4; this is schematically illustrated in figure 2. This four-stroke quantum

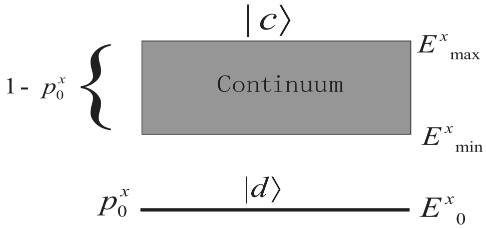

Figure 1. An illustration of the level structure of the working medium. The occupation probability $p_0^x, x = l, h$ of the discrete level $|d\rangle$ (with eigenenergy E_0^x) was kept fixed in adiabatic processes. The continuum broadening was denoted by $E^x_{\max} - E^x_{\min}$.

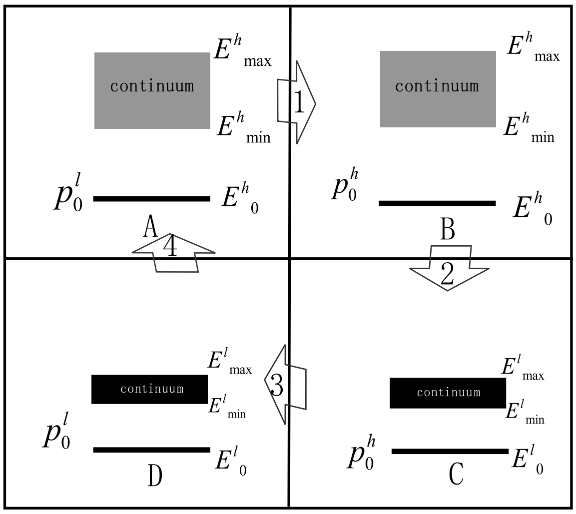

Figure 2. Schematic illustration of the four-stroke quantum heat engine. From states A to B, the working medium absorbs heat from the high-temperature reservoir, leading to population transfer from the discrete level $|d\rangle$ to the continuum. From states B to C, works are done with the working medium undergoing a quasi-adiabatic process. Stages 3 and 4 (corresponding changes from C to D and from D to A, respectively.) are reversed processes of 1 and 2, respectively. We will use $(E_0^x, E^x_{\min}, E^x_{\max})$ to characterize the level structure of the working medium in the text.

heat engine is a quantum analogue of the classical Otto engine, which includes two quantum adiabatic processes and two isothermal processes.

Denoting $p_0^x, x = h, l$, the occupation probability of the discrete level and $p(E)$ the occupation probability of the continuum, we can write the expectation value of the measured energy of a quantum system U as

$$U = \langle E\rangle = \sum_i p_i E_i + \int \mathrm{d}[p(E)\cdot E]. \tag{17}$$

The definition of infinitesimal work done in a process is then

$$đW = \sum_i p_i\, \mathrm{d}E_i + \sum_{E_{\min}}^{E_{\max}} p(E)\, \mathrm{d}E, \tag{18}$$

which is a straightforward extension of that for discrete level systems [13] to the system under our consideration. The first term comes from the contribution of discrete levels, while the last term comes from the continuum. By the first law of thermodynamics $\mathrm{d}U = đQ + đW$, the infinitesimal heat absorbed is

$$đQ = \sum_i E_i \,\mathrm{d}p_i + \sum_{E_{\min}}^{E_{\max}} E \,\mathrm{d}p(E). \tag{19}$$

With these notations, we now calculate the work done on the four stages of the heat engine.

In stage 1, namely, from A to B, the working medium is coupled to a hot reservoir of temperature T_h and its energy structure is kept fixed. In this process, the population of the discrete level is changing from the initial population p_0^l to the population p_0^h. Accordingly, the total population of the continuum is changing from $1 - p_0^l$ to $1 - p_0^h$. The work done in this stage is clearly zero by the definition equation (18). In stage 2, B $\to$ C, the working medium is decoupled from the heat reservoir, and the energy structure is varied from $\left(E_0^h, E_{\min}^h, E_{\max}^h\right)$ to $\left(E_0^l, E_{\min}^l, E_{\max}^l\right)$. In this process, the occupation probability p_0^h is kept fixed. This is a quasi-adiabatic process in the sense that the total occupation probability of the working medium on the continuum remains unchanged, but population transfer among states in the continuum is allowed. Physically, this can be realized in the same way as we discussed in the last section. After the working medium reaches thermodynamical equilibrium, the total occupation probabilities of the working medium on the continuum satisfy

$$\begin{aligned}
1 - p_0^h &= \int_{E_{\min}^h}^{E_{\max}^h} \frac{\rho_h}{Z_{hh}} \,\mathrm{e}^{-\beta_h E^h} \,\mathrm{d}E^h, \\
1 - p_0^h &= \int_{E_{\min}^l}^{E_{\max}^l} \frac{\rho_l}{Z_{hl}} \,\mathrm{e}^{-\beta_h E^l} \,\mathrm{d}E^l \\
&= \int_{E_{\min}^h}^{E_{\max}^h} \frac{\rho_h}{Z_{hl}} \,\mathrm{e}^{-\beta_h \left(\frac{\rho_h}{\rho_l}\left(E^h - E_{\min}^h\right) + E_{\min}^l\right)} \,\mathrm{d}E^h, \\
1 - p_0^l &= \int_{E_{\min}^l}^{E_{\max}^l} \frac{\rho_l}{Z_{ll}} \,\mathrm{e}^{-\beta_l E^l} \,\mathrm{d}E^l \\
&= \int_{E_{\min}^h}^{E_{\max}^h} \frac{\rho_h}{Z_{ll}} \,\mathrm{e}^{-\beta_l \left(\frac{\rho_h}{\rho_l}\left(E^h - E_{\min}^h\right) + E_{\min}^l\right)} \,\mathrm{d}E^h, \\
1 - p_0^l &= \int_{E_{\min}^h}^{E_{\max}^h} \frac{\rho_h}{Z_{lh}} \,\mathrm{e}^{-\beta_l E^h} \,\mathrm{d}E^h.
\end{aligned} \tag{20}$$

Here ρ_h (ρ_l) denotes the degeneracy of the continuum with the level structure $\left(E_0^h, E_{\min}^h, E_{\max}^h\right)\left(E_0^l, E_{\min}^l, E_{\max}^l\right)$, and it is assumed to be constant. $E^x, x = h, l$ stand for an eigenenergy in the continuum with the level structure $\left(E_0^x, E_{\min}^x, E_{\max}^x\right)$. Z_{hh}, Z_{hl}, Z_{ll} and Z_{lh} are the partition functions of the working medium at equilibrium states B, C, D and A, respectively. $\beta_h = \frac{1}{KT_h}, \beta_l = \frac{1}{KT_l}$, and K is the Boltzmann constant. We have assumed in equation (20) that the occupation probability on the continuum satisfies the second line of equation (20) after stage 2. This is reachable in a non-unitary evolution by changing the parameters upon which the energy structure of the working medium depends. Throughout this paper, we focus our attention on the following situation:

$$\rho_h\left(E_{\max}^h - E_{\min}^h\right) = \rho_l\left(E_{\max}^l - E_{\min}^l\right), \qquad \rho_h\left(E^h - E_{\min}^h\right) = \rho_l\left(E^l - E_{\min}^l\right). \tag{21}$$

These relations mean that the quasi-adiabatic process does not change the distribution of microstates. In other words, the degeneracy of the continuum is supposed to be changed homogeneously in adiabatic processes, and then any state with energy E^h at temperature T^h

has a one-valued correspondence E^l at temperature T^l. In fact, this relation was used in third and fifth lines in equation (20).

Define

$$E^h_{\min} - E^h_0 = \Delta^h, \qquad E^h_{\max} - E^h_{\min} = \delta^h$$

$$E^l_{\min} - E^l_0 = \Delta^l, \qquad E^l_{\max} - E^l_{\min} = \delta^l.$$

The energy change in stage 2 reads

$$\begin{aligned}\Delta U_2 &= p^h_0\left(E^h_0 - E^l_0\right) + \int_{E^h_{\min}}^{E^h_{\max}} \frac{\rho_h}{Z_{hh}}\, \mathrm{e}^{-\beta_h E^h} E^h\, \mathrm{d}E^h - \int_{E^l_{\min}}^{E^l_{\max}} \frac{\rho_l}{Z_{hl}}\, \mathrm{e}^{-\beta_h E^l} E^l\, \mathrm{d}E^l \\ &= p^h_0\left(E^h_0 - E^l_0\right) + \rho_h \int_{E^h_{\min}}^{E^h_{\max}} \frac{1}{Z_{hh}}\, \mathrm{e}^{-\beta_h E^h} \left(E^h - \left(\frac{\rho_h}{\rho_l}\left(E^h - E^h_{\min}\right) + E^l_{\min}\right)\right) \mathrm{d}E^h \\ &\quad + \rho_h \int_{E^h_{\min}}^{E^h_{\max}} \left(\frac{1}{Z_{hh}}\, \mathrm{e}^{-\beta_h E^h} - \frac{1}{Z_{hl}}\, \mathrm{e}^{-\beta_h \left(\frac{\rho_h}{\rho_l}\left(E^h - E^h_{\min}\right) + E^l_{\min}\right)}\right)\left(\frac{\rho_h}{\rho l}\left(E^h - E^h_{\min}\right) + E^l_{\min}\right) \mathrm{d}E^h.\end{aligned} \tag{22}$$

According to equation (18), the work done in this stage reads

$$\begin{aligned}\Delta W_2 &= p^h_0\left(E^h_0 - E^l_0\right) + \int_{E^h_{\min}}^{E^h_{\max}} \frac{\rho_h}{Z_{hh}}\, \mathrm{e}^{-\beta_h E^h} \left(E^h - \left(\frac{\rho_h}{\rho_l}\left(E^h - E^h_{\min}\right) + E^l_{\min}\right)\right) \mathrm{d}E^h \\ &= p^h_0\left(E^h_0 - E^l_0\right) + \left(1 - p^h_0\right) \frac{\rho_h E^h_{\min} - \rho_l E^l_{\min}}{\rho_l} \\ &\quad + \left(1 - p^h_0\right)\left(\frac{\rho_l - \rho_h}{\rho_l}\left(E^h_{\max} + \frac{\delta^h}{\mathrm{e}^{-\beta_h \delta^h} - 1}\right) + \frac{\rho_l - \rho_h}{\beta_h \rho_l}\right),\end{aligned} \tag{23}$$

which is exactly the second line in equation (22). Stage 3 is similar to the first. The working medium is now coupled to a cold reservoir at temperature T_l, and its energy structure is kept fixed. The occupation probability backs in this stage from p^h_0 to p^l_0. Stage 4 closes the cycle and is similar to stage 2. The working medium is decoupled from the cold reservoir, and the level structure is changed back to its original value $\left(E^h_0, E^h_{\min}, E^h_{\max}\right)$. A similar analysis shows that the work done in stage 3 is zero, whereas the energy change and the work done in stage 4 are

$$\begin{aligned}-\Delta U_4 &= p^h_0\left(E^h_0 - E^l_0\right) + \int_{E^h_{\min}}^{E^h_{\max}} \frac{\rho_h}{Z_{lh}}\, \mathrm{e}^{-\beta_l E^h} E^h\, \mathrm{d}E^h - \int_{E^l_{\min}}^{E^l_{\max}} \frac{\rho_l}{Z_{ll}}\, \mathrm{e}^{-\beta_l E^l} E^l\, \mathrm{d}E^l \\ &= p^l_0\left(E^h_0 - E^l_0\right) + \rho_h \int_{E^h_{\min}}^{E^h_{\max}} \left(\frac{1}{Z_{lh}}\, \mathrm{e}^{-\beta_l E^h} - \frac{1}{Z_{ll}}\, \mathrm{e}^{-\beta_l \left(\frac{\rho_h}{\rho_l}\left(E^h - E^h_{\min}\right) + E^l_{\min}\right)}\right) E^h\, \mathrm{d}E^h \\ &\quad + \rho_h \int_{E^h_{\min}}^{E^h_{\max}} \frac{1}{Z_{ll}}\, \mathrm{e}^{-\beta_l \left(\frac{\rho_h}{\rho_l}\left(E^h - E^h_{\min}\right) + E^l_{\min}\right)} \left(E^h - \left(\frac{\rho_h}{\rho_l}\left(E^h - E^h_{\min}\right) + E^l_{\min}\right)\right) \mathrm{d}E^h,\end{aligned} \tag{24}$$

$$\begin{aligned}-\Delta W_4 &= p^l_0\left(E^h_0 - E^l_0\right) + \rho_h \int_{E^h_{\min}}^{E^h_{\max}} \frac{1}{Z_{ll}}\, \mathrm{e}^{-\beta_l \left(\frac{\rho_h}{\rho_l}\left(E^h - E^h_{\min}\right) + E^l_{\min}\right)} \\ &\quad \times \left(E^h - \left(\frac{\rho_h}{\rho_l}\left(E^h - E^h_{\min}\right) + E^l_{\min}\right)\right) \mathrm{d}E^h\end{aligned}$$

$$= p_0^l\left(E_0^h - E_0^l\right) + \left(1 - p_0^l\right)\frac{\rho_h E_{\min}^h - \rho_l E_{\min}^l}{\rho_l} + \left(1 - p_0^l\right)\left(\frac{\rho_l - \rho_h}{\rho_l}\left(E_{\max}^h + \frac{\delta^h}{\mathrm{e}^{-\beta_l \frac{\rho_h}{\rho_l}\delta^h} - 1}\right) + \frac{\rho_l - \rho_h}{\beta_l \rho_h}\right). \tag{25}$$

The net work done in the whole cycle is then

$$\Delta W = \Delta W_2 + \Delta W_4 = \left(p_0^h - p_0^l\right)\left(E_0^h - E_0^l\right) + \left(p_0^l - p_0^h\right)\frac{\rho_h E_{\min}^h - \rho_l E_{\min}^l}{\rho_l} + \left(p_0^l - p_0^h\right)\frac{\rho_l - \rho_h}{\rho_l}E_{\max}^h + \frac{\rho_l - \rho_h}{\rho_l}\delta^h\left(\frac{1 - p_0^h}{\mathrm{e}^{-\beta_h \delta^h} - 1} - \frac{1 - p_0^l}{\mathrm{e}^{-\beta_l \frac{\rho_h}{\rho_l}\delta^h} - 1}\right) + (\rho_l - \rho_h)\left(\frac{1 - p_0^h}{\beta_h \rho_l} - \frac{1 - p_0^l}{\beta_l \rho_h}\right). \tag{26}$$

Noting

$$\frac{\rho_l}{\rho_h} = \frac{\delta^h}{\delta^l}, \tag{27}$$

one can reduce the net work ΔW to

$$\Delta W = \left(p_0^l - p_0^h\right)((\Delta^h + \delta^h) - (\Delta^l + \delta^l)) + (\delta^h - \delta^l) \cdot f\left(p_0^x, T_x, \delta^x\right)\big|_{x=l,h}. \tag{28}$$

Here

$$f\left(p_0^x, T_x, \delta^x\right)\big|_{x=l,h} = \frac{1 - p_0^h}{\mathrm{e}^{-\frac{\delta^h}{KT_h}} - 1} - \frac{1 - p_0^l}{\mathrm{e}^{-\frac{\delta^l}{KT_l}} - 1} + \frac{1 - p_0^h}{\frac{\delta^h}{KT_h}} - \frac{1 - p_0^l}{\frac{\delta^l}{KT_l}}. \tag{29}$$

This is the central result of this paper, showing that the net work done by the heat engine depends on the occupation probabilities p_0^h and p_0^l, the continuum broadenings δ^h and δ^l, the energy gaps Δ^h and Δ^l as well as the low and high temperatures of the reservoir. To get more insight into this result, we consider the following limiting situations. (a) The continuum broadening remains unchanged in the cycle, namely, $\delta^h = \delta^l$. The net work in this case reads

$$\Delta W_\delta = \left(p_0^l - p_0^h\right)(\Delta^h - \Delta^l). \tag{30}$$

This backs to the net work done by the quantum heat engine with two-level systems as its working medium. (b) High-temperature limit, $\frac{\delta^h}{KT_h} \ll 1$, $\frac{\delta^l}{KT_l} \ll 1$. The net work done in this situation follows

$$\Delta W_T = \left(p_0^l - p_0^h\right)((\Delta^h + \delta^h) - (\Delta^l + \delta^l)). \tag{31}$$

Interestingly, the net work in this case takes the same form as in equation (30), but the energy difference of the two-level working medium is $(\Delta^h + \delta^h)$ at high temperature and $(\Delta^l + \delta^l)$ at low temperature. The same results are found in the low-temperature limit. (c) No population transfer between the discrete level and the continuum in the cycle, i.e. $p_0^h = p_0^l = p$. The net work ΔW in this case follows from equation (28):

$$\Delta W_p = (\delta^h - \delta^l)(1 - p) f\left(p_0^x = 0, T_x, \delta^x\right)\big|_{x=l,h}. \tag{32}$$

As shown, ΔW_p totally comes from the contribution of the continuum. It is zero if $\delta^h = \delta^l$, and it increases linearly as p decreases. $\Delta W_p > 0$ requires that $T_h > \frac{\delta^h}{\delta^l}T_l$ and $\delta^h > \delta^l$. This is similar to the requirement upon the two-level quantum heat engine [1, 19] for positive work extraction. In order to compare our heat engine with the two-level one, we plot a work difference $(\Delta W - \Delta W_\delta)$ versus δ^h and δ^l in figure 3. Note that this work difference is different

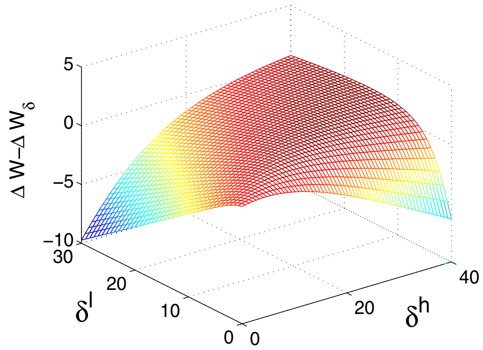

Figure 3. The work difference $(\Delta W - \Delta W_\delta)$ as a function of δ^h and δ^l. The parameters chosen are $p_0^l = 0.5$, $p_0^h = 0.3$ and $KT_h = 5$. The work difference, δ^x, $(x = l, h)$, and KT_h were rescaled in units of $KT_l = 1$ in this plot.

(This figure is in colour only in the electronic version)

from ΔW_p, where $p_0^l = p_0^h = p$ is considered. As we mentioned above, the contribution from the continuum was excluded in ΔW_δ. So $(\Delta W - \Delta W_\delta)$ mostly characterize the effect of the continuum on the work extraction in the quantum heat engine. From the other aspect, this work difference can be understood as the net work with the energy gap unchanged in the cycle, i.e. $\Delta^l = \Delta^h$. Figure 3 shows that the work difference decreases as δ^l increases for small δ^h, but the result goes in the opposite direction for large δ^h. We also find from figure 3 that $(\Delta W - \Delta W_\delta) > 0$ in the region $\delta^l \simeq \delta^h$ and around.

Using the definition of the heat engine efficiency,

$$\eta = 1 - \frac{\Delta Q_2 + \Delta Q_3}{\Delta Q_1 + \Delta Q_4}, \tag{33}$$

and

$$\begin{aligned} \Delta Q_1 + \Delta Q_4 &= \left(p_0^l - p_0^h\right)(\Delta^h + \delta^h) + \delta^h \cdot f\left(p_0^x, T_x, \delta^x\right)\big|_{x=l,h}, \\ \Delta Q_2 + \Delta Q_3 &= \left(p_0^l - p_0^h\right)(\Delta^l + \delta^l) + \delta^l \cdot f\left(p_0^x, T_x, \delta^x\right)\big|_{x=l,h}, \end{aligned} \tag{34}$$

we have

$$\eta = 1 - \frac{\left(p_0^l - p_0^h\right)(\Delta^l + \delta^l) + \delta^l \cdot f\left(p_0^x, T_x, \delta^x\right)\big|_{x=l,h}}{\left(p_0^l - p_0^h\right)(\Delta^h + \delta^h) + \delta^h \cdot f\left(p_0^x, T_x, \delta^x\right)\big|_{x=l,h}}. \tag{35}$$

In the high-temperature limit $\frac{\delta^h}{KT_h} \ll 1$, $\frac{\delta^l}{KT_l} \ll 1$, η reduces to

$$\eta = 1 - \frac{\Delta^l + \delta^l}{\Delta^h + \delta^h}, \tag{36}$$

returning back to the efficiency of the two-level quantum heat engine. This observation holds in the low temperature, as the net work does. Similarly, for $p_0^l = p_0^h$, the efficiency becomes $\eta = 1 - \delta^l/\delta^h$. Note that in the limit $\delta^l = \delta^h$, the net work ΔW returns back to the result of the two-level quantum heat engine, but the efficiency does not. This is due to the difference in the heat exchange of the two engines.

4. Conclusion and remark

In conclusion, a new kind of quantum heat engine has been introduced in this paper. As its working medium, the three-level quantum system and the quantum system that has a discrete level and a continuum are considered. The working medium is allowed to undergo quantum quasi-adiabatic and non-unitary evolutions in the cycle. This makes the engine different from the conventional quantum heat engines. This new quantum heat engine can extract work like a two-level quantum heat engine in the high-temperature and low-temperature limits, whereas it works in a different way at intermediate temperatures. Since the previous studies on quantum heat engine were focused on various working media only with discrete energy levels, the study presented here can better the understanding of quantum heat engine, in particular shed light on the effect of non-adiabatic and non-unitary evolution on the performance of quantum heat engine. The limitation of our discussion is the assumption on the occupation probability in the excited states (or the continuum) after stage 2. However, it is reachable by manipulating the population transfer in the quasi-adiabatic process via controlling the parameters, upon which the energy structure of the working medium depends. The presented representation can be readily extended to other non-adiabatic processes.

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